

A Note on a result due to Ankeny and Rivlin¹

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Abstract. Let $p(z) = a_0 + a_1z + a_2z^2 + a_3z^3 + \cdots + a_nz^n$ be a polynomial of degree n having no zeros in the unit disk. Then it is well known that for $R \geq 1$, $\max_{|z|=R} |p(z)| \leq \left(\frac{R^n + 1}{2}\right) \max_{|z|=1} |p(z)|$. In this paper, we consider polynomials with gaps, having all its zeros on the circle $S(0, K) := \{z : |z| = K\}$, $0 < K \leq 1$, and estimate the value of $\left(\frac{\max_{|z|=R} |p(z)|}{\max_{|z|=1} |p(z)|}\right)^s$ for any positive integer s .

Keywords: Inequalities, polynomials, Zeros.

1. INTRODUCTION

Let $p(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n . We will denote

$$M(p, r) := \max_{|z|=r} |p(z)|, \quad r > 0,$$

$$||p|| := \max_{|z|=1} |p(z)|,$$

and

$$D(0, K) := \{z : |z| < K\}, \quad K > 0.$$

Bernstein observed the following result, which in fact is a simple consequence of the maximum modulus principle (see [8, p. 137]). This inequality is also known as the Bernstein's inequality.

Theorem 1.1. *Let $p(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n . Then for $R \geq 1$,*

$$M(p, R) \leq R^n ||p||. \quad (1.1)$$

Equality holds for $p(z) = \alpha z^n$, α being a complex number.

¹This is a preprint of a paper whose final and definite form is published open access in Applied Mathematics E-Notes. See <http://www.math.nthu.edu.tw/~amen/> for the final version.

For polynomial of degree n not vanishing in the interior of the unit circle, Ankeny and Rivlin [1] proved the following result.

Theorem 1.2 (Ankeny and Rivlin [1]). *Let $p(z) = \sum_{j=0}^n a_j z^j \neq 0$ in $D(0, 1)$. Then for $R \geq 1$,*

$$M(p, R) \leq \left(\frac{R^n + 1}{2} \right) \|p\|. \quad (1.2)$$

Here equality holds for $p(z) = \frac{\alpha + \beta z^n}{2}$, where $|\alpha| = |\beta| = 1$.

In 2005, Gardner, Govil and Musukula [3] proved the following generalization and sharpening of Theorem 1.2.

Theorem 1.3. *Let $p(z) = a_0 + \sum_{j=t}^n a_j z^j$, $1 \leq t \leq n$, be a polynomial of degree n and $p(z) \neq 0$ in $D(0, K)$, $K \geq 1$. Then for $R \geq 1$,*

$$M(p, R) \leq \left(\frac{R^n + s_0}{1 + s_0} \right) \|p\| - \left(\frac{R^n - 1}{1 + s_0} \right) m - \frac{n}{1 + s_0} \left[\frac{(\|p\| - m)^2 - (1 + s_0)^2 |a_n|^2}{(\|p\| - m)} \right] \quad (1.3)$$

$$\times \left\{ \frac{(R - 1)(\|p\| - m)}{(\|p\| - m) + (1 + s_0)|a_n|} - \ln \left[1 + \frac{(R - 1)(\|p\| - m)}{(\|p\| - m) + (1 + s_0)|a_n|} \right] \right\},$$

where $m = \min_{|z|=K} |p(z)|$, and

$$s_0 = K^{t+1} \frac{\frac{t}{n} \cdot \frac{|a_t|}{|a_0| - m} K^{t-1} + 1}{\frac{t}{n} \cdot \frac{|a_t|}{|a_0| - m} K^{t+1} + 1}.$$

Several research monographs have been written on this subject of inequalities (see for example Govil and Mohapatra [4], Milovanović, Mitrinović and Rassias [7], Rahman and Schmeisser [9], and recent article of Govil and Nwaeze [5]).

While trying to obtain an inequality analogous to (1.2) for polynomials not vanishing in $D(0, K)$, $K \leq 1$, Dewan and Ahuja [2] were able to prove this only for polynomials having all the zeros on the circle $S(0, K) := \{z : |z| = K\}$, $0 < K \leq 1$.

Theorem 1.4. *Let $p(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n having all its zeros on $S(0, K)$, $K \leq 1$. Then for $R \geq 1$ and for every positive integer s ,*

$$\{M(p, R)\}^s \leq \left[\frac{K^{n-1}(1+K) + (R^{ns} - 1)}{K^{n-1} + K^n} \right] \{M(p, 1)\}^s. \quad (1.4)$$

For $s = 1$, the Theorem 1.4 reduces to

Corollary 1.5. *Let $p(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n having all its zeros on $S(0, K)$, $K \leq 1$. Then for $R \geq 1$,*

$$M(p, R) \leq \left[\frac{K^{n-1}(1+K) + (R^n - 1)}{K^{n-1} + K^n} \right] M(p, 1). \quad (1.5)$$

In same spirit, we prove the following results

2. MAIN RESULTS

Theorem 2.1. *Let $p(z) = z^m \left[a_{n-m} z^{n-m} + \sum_{j=\mu}^{n-m} a_{n-m-j} z^{n-m-j} \right]$, $1 \leq \mu \leq n-m$, $0 \leq m \leq n-1$, be a polynomial of degree n , having m -fold zeros at origin and remaining $n-m$ zeros on $S(0, K)$, $K \leq 1$. Then for $R \geq 1$ and every positive integer s ,*

$$[M(p, R)]^s \leq L(\mu; K, m, n, s) [M(p, 1)]^s, \quad (2.1)$$

where

$$L(\mu; K, m, n, s) = \frac{n(K^{n-m-2\mu+1} + K^{n-m-\mu+1}) + (R^{ns} - 1)[n + mK^{n-m-2\mu+1} + mK^{n-m-\mu+1} - m]}{n(K^{n-m-2\mu+1} + K^{n-m-\mu+1})}.$$

For $m = 0$, we have

Corollary 2.2. *Let $p(z) = a_n z^n + \sum_{j=\mu}^n a_{n-j} z^{n-j}$, $1 \leq \mu \leq n$, be a polynomial of degree n , having all zeros on $|z| = K$, $K \leq 1$. Then for $R \geq 1$ and every positive integer s ,*

$$[M(p, R)]^s \leq L(\mu; K, n, s) [M(p, 1)]^s, \quad (2.2)$$

where

$$L(\mu; K, n, s) = \frac{K^{n-\mu}(K^{1-\mu} + K) + (R^{ns} - 1)}{K^{n-2\mu+1} + K^{n-\mu+1}}.$$

If we set $\mu = 1$ into Corollary 2.2, we get the following result of Dewan and Ahuja [2].

Corollary 2.3. Let $p(z) = \sum_{j=0}^n a_j z^j$, be a polynomial of degree n , having all zeros on $|z| = K$, $K \leq 1$. Then for $R \geq 1$ and every positive integer s ,

$$[M(p, R)]^s \leq L(1; K, n, s)[M(p, 1)]^s, \quad (2.3)$$

where

$$L(1; K, n, s) = \frac{K^{n-1}(1+K) + (R^{ns} - 1)}{K^{n-1} + K^n}.$$

3. LEMMAS

For the proof Theorem 2.1 we need the following lemmas. The first lemma is due to Kumar and Lal [6].

Lemma 3.1. Let $p(z) = z^m \left[a_{n-m} z^{n-m} + \sum_{j=\mu}^{n-m} a_{n-m-j} z^{n-m-j} \right]$, $1 \leq \mu \leq n-m$, $0 \leq m \leq n-1$, be a polynomial of degree n , having m -fold zeros at origin and remaining $n-m$ zeros on $|z| = K$, $K \leq 1$.

$$\max_{|z|=1} |p'(z)| \leq \frac{n + m(K^{n-m-2\mu+1} + K^{n-m-\mu+1} - 1)}{K^{n-m-2\mu+1} + K^{n-m-\mu+1}} \max_{|z|=1} |p(z)|. \quad (3.1)$$

The next lemma is the Bernstein inequality given in Theorem 1.1.

Lemma 3.2. Let $p(z)$ be a polynomial of degree n . Then for $R \geq 1$,

$$M(p, R) \leq R^n M(p, 1). \quad (3.2)$$

4. PROOF

Proof of Theorem 2.1. By Lemma 3.1, we have

$$\max_{|z|=1} |p'(z)| \leq \frac{n + m(K^{n-m-2\mu+1} + K^{n-m-\mu+1} - 1)}{K^{n-m-2\mu+1} + K^{n-m-\mu+1}} \max_{|z|=1} |p(z)|.$$

Applying Lemma 3.2 to the polynomial $p'(z)$ which is of degree $n-1$, it follows that for all $R \geq 1$ and $\theta \in [0, 2\pi)$,

$$\begin{aligned} |p'(Re^{i\theta})| &\leq \max_{|z|=R} |p'(z)| \\ &\leq R^{n-1} \max_{|z|=1} |p'(z)| \\ &\leq R^{n-1} \left[\frac{n + m(K^{n-m-2\mu+1} + K^{n-m-\mu+1} - 1)}{K^{n-m-2\mu+1} + K^{n-m-\mu+1}} \right] \max_{|z|=1} |p(z)|. \end{aligned}$$

So for each $\theta \in [0, 2\pi)$ and $R \geq 1$, we obtain

$$\begin{aligned}
[p(Re^{i\theta})]^s - [p(e^{i\theta})]^s &= \int_1^R \frac{d[p(te^{i\theta})]^s}{dt} dt \\
&= \int_1^R s[p(te^{i\theta})]^{s-1} p'(te^{i\theta}) e^{i\theta} dt.
\end{aligned}$$

This implies that

$$|p(Re^{i\theta})|^s \leq |p(e^{i\theta})|^s + s \int_1^R |p(te^{i\theta})|^{s-1} |p'(te^{i\theta})| dt.$$

So,

$$\begin{aligned}
[M(p, R)]^s &\leq [M(p, 1)]^s + s \int_1^R [t^n M(p, 1)]^{s-1} |p'(te^{i\theta})| dt \\
&\leq [M(p, 1)]^s + s \int_1^R t^{ns-n} [M(p, 1)]^{s-1} t^{n-1} \frac{n + m(K^{n-m-2\mu+1} + K^{n-m-\mu+1} - 1)}{K^{n-m-2\mu+1} + K^{n-m-\mu+1}} M(p, 1) dt \\
&= [M(p, 1)]^s + s \left[\frac{n + m(K^{n-m-2\mu+1} + K^{n-m-\mu+1} - 1)}{K^{n-m-2\mu+1} + K^{n-m-\mu+1}} \right] [M(p, 1)]^s \int_1^R t^{ns-1} dt \\
&= [M(p, 1)]^s + [M(p, 1)]^s \left[\frac{n + m(K^{n-m-2\mu+1} + K^{n-m-\mu+1} - 1)}{K^{n-m-2\mu+1} + K^{n-m-\mu+1}} \right] s \frac{R^{ns} - 1}{ns} \\
&= [M(p, 1)]^s \left[1 + \frac{[n + m(K^{n-m-2\mu+1} + K^{n-m-\mu+1} - 1)] (R^{ns} - 1)}{n(K^{n-m-2\mu+1} + K^{n-m-\mu+1})} \right].
\end{aligned}$$

This yields

$$[M(p, R)]^s \leq [M(p, 1)]^s \left[\frac{n(K^{n-m-2\mu+1} + K^{n-m-\mu+1}) + [n + m(K^{n-m-2\mu+1} + K^{n-m-\mu+1} - 1)] (R^{ns} - 1)}{n(K^{n-m-2\mu+1} + K^{n-m-\mu+1})} \right].$$

This completes the proof. □

Acknowledgment. Many thanks to the anonymous referee for his/her valuable comments.

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